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LOSS OF STABILITY IN HAMILTONIAN SYSTEMS THAT DEPEND ON PARAMETERS†

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The phenomenon of soft and hard loss of stability of equilibrium positions and periodic trajectories in typical one-parameter families of Hamiltonian systems is investigated.

1. THE POSITION OF EQUILIBRIUM

THE SPECTRUM S of the linearization matrix contains the basic information about the stability of a position of equilibrium of a system. In Hamiltonian systems the set $S \subset \mathbf{C}$ is symmetrical about the real and imaginary axes. The elements of S are the roots of the characteristic polynomial $f(\lambda^2)$ and are known as the characteristic values or eigenvalues of the position of equilibrium.

Suppose we have a one-parameter family of autonomous Hamiltonian systems with n degrees of freedom, with phase space M and Hamiltonians H_ϵ; ϵ is a parameter taking values in the neighbourhood of zero. Let $x(\epsilon) \in M$ be a smooth family of positions of equilibrium. Assume that the eigenvalues of the position of equilibrium $x(0)$ are $\pm i\omega_1, \ldots, \pm i\omega_n$, where $\omega_1, \ldots, \omega_n$ are real and non-zero. We shall assume further that $\omega_1 = \omega_2 = \omega$ and that the frequencies $|\omega|, |\omega_3|, \ldots, |\omega_n|$ are pairwise distinct; as ϵ goes through zero two pairs of imaginary eigenvalues of the position of equilibrium $x(\epsilon)$ "collide" at the points $\pm i\omega$ and begin to form a quadruple $\pm\alpha \pm i\beta$, $\alpha\beta \neq 0$. We note that a necessary condition for this bifurcation to occur is that the Jordan form of the linearization matrix of the system at $x(0)$ contains a pair of Jordan blocks.

Under the above assumptions, if ϵ is negative but small in absolute value, the position of equilibrium $x(\epsilon)$ will be stable in the linear approximation. As the parameter ϵ goes through zero, however, it will lose its stability. It can be shown that in typical one-parameter families of Hamiltonian systems a position of equilibrium cannot lose its stability in any other way.

One may think that a position of equilibrium will destabilize if two imaginary eigenvalues $\pm i\omega$ collide at zero and then diverge along the real axis. However, bifurcation of this kind does not take place as a rule, because at the time of collision the position of equilibrium collides with another position of equilibrium, becomes degenerate and, when the parameter varies further, disappears [1]. The position of equilibrium may fail to disappear if the system possesses a certain symmetry.

Any relation of the form

$$k\omega + k_3\omega_3 + \ldots + k_n\omega_n = 0$$

where $k, k_3, \ldots, k_n$ are integers, will be called a resonance of order $|k| + |k_3| + \ldots + |k_n|$.

Let D_ϵ be the discriminant of the characteristic polynomial $f_\epsilon(\lambda^2)$ of the position of equilibrium $x(\epsilon)$, and $U_{\epsilon,\alpha}$ $(\alpha > 0)$ the family of neighbourhoods

$$U_{\epsilon,\alpha} = \{x \in M: \ \mathrm{dist}(x, x(\epsilon)) < \alpha\}$$

where the distance is understood in some Riemannian metric on M.

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Lemma 1. Assume that the frequencies $\omega, \omega_3, \ldots, \omega_n$ satisfy no resonances of orders less than or equal to four, and that $dD_\epsilon/d\epsilon|_{\epsilon=0} \neq 0$. Then, for every ϵ in some neighbourhood of zero, there is a neighbourhood of $x(\epsilon)$ in M in which one can introduce local coordinates p, q, y, x; $p = (p_1, p_2)$, $q = (q_1, q_2)$, $y = (y_1, \ldots, y_{n-2})$, $x = (x_1, \ldots, x_{n-2})$ such that:

(a) The pairs of variables p and q; y and x are canonically conjugate;

(b) The coordinates of $x(\epsilon)$ are zero;

(c) The following equality holds:

$$H_\epsilon = G(p, q, \epsilon) + \sum_{k=1}^{n-2} G_k(y_k, x_k, \epsilon) + G'(p, q, y, x, \epsilon) \tag{1.1}$$

where

$$\begin{aligned} G &= (\omega + \epsilon\omega'(\epsilon))(p_2 q_1 - p_1 q_2) + 2p^2 - \epsilon g(\epsilon) q^2/2 + \\ &+ q^2[Aq^2 + B(p_2 q_1 - p_1 q_2) + Cp^2], \quad g(0) > 0 \\ G_k &= (\omega_{k+2} + \epsilon\omega'_{k+2}(\epsilon))(y_k^2 + x_k^2)/2 + A_k(y_k^2 + x_k^2)^2 \end{aligned} \tag{1.2}$$

$$G' = \epsilon O_4 + O_5, \quad O_l = O_l(p, q, y, x); \quad p^2 = p_1^2 + p_2^2, \quad q^2 = q_1^2 + q_2^2 \tag{1.3}$$

(O_l is a function whose Maclaurin series begins with terms of degree at least l).

Remarks. 1. When $n = 2$ the functions G_k and variables y, x do not appear in H_ϵ.

2. The assumptions of Lemma 1, and also the inequalities $A \neq 0$, $A_k \neq 0$, $1 \leq k \leq n-2$, are the conditions of general position; that is, they are satisfied by typical one-parameter families H_ϵ.

3. The eigenvalues of the position of equilibrium $x(\epsilon)$ are

$$\begin{aligned} &\pm i\omega_1(\epsilon), \ldots, \pm i\omega_n(\epsilon) \\ &\omega_k^2(\epsilon) = (\omega + \epsilon\omega')^2 \pm [-8\epsilon g(\epsilon)(\omega + \epsilon\omega')^2 - 16\epsilon^2 g^2(\epsilon)]^{1/2}, \quad k = 1, 2 \\ &\omega_k(\epsilon) = \omega_k + \epsilon\omega'_k(\epsilon), \quad k = 3, \ldots, n \end{aligned}$$

4. There is no loss of generality in replacing A by $\operatorname{sgn} A/8$.

5. If $n = 2$ and $A \neq 0$ then, for negative ϵ of small absolute value, the positions of equilibrium $x(\epsilon)$ are Lyapunov stable.

6. If $n > 2$, $A \neq 0$ and $A_k \neq 0$, $k = 1, \ldots, n-2$, then, for negative ϵ of small absolute value, the positions of equilibrium $x(\epsilon)$ are metrically stable, in the following sense: for any $\beta > 0$ there exists $\alpha > 0$ such that a solution with initial data in a set $V_{\epsilon,\alpha} \subset U_{\epsilon,\alpha}$ will always remain in the neighbourhood $U_{\epsilon,\beta}$, and moreover

$$\lim_{\beta \to 0} \mu(V_{\epsilon,\alpha})/\mu(U_{\epsilon,\alpha}) = 1$$

where $\mu(W)$ denotes the measure of a set W.

Our main theorem on the loss of stability in a position of equilibrium may be stated as follows.

Theorem 1. Under the assumptions of Lemma 1, suppose that as the parameter ϵ is varied an equilibrium position of an analytic Hamiltonian system loses stability in accordance with the scenario described above. Then the following statements are true:

1. If the constant A in formula (1.2) is negative then, as ϵ goes through zero, the system undergoes a hard loss of (metric) stability: for any small $\epsilon \geq 0$ there exists a set $G_\epsilon^u \subset M$ such that, for any $\alpha > 0$, solutions with initial data in $G_\epsilon^u \cap U_{\epsilon\alpha}$ will leave a neighbourhood U_{ϵ,α_0}, where $\alpha_0 > 0$ is independent of ϵ, α, and for $\alpha < c^*$

$$\mu(G_\epsilon^u \cap U_{\epsilon,\alpha})/\mu(U_{\epsilon,\alpha}) \geq c > 0 \tag{1.4}$$

the constants c, c^* being independent of ϵ.

2. If $A > 0$ and $n = 2$ then, as ϵ goes through zero, the system undergoes a soft loss of stability: for small $\epsilon \geq 0$, solutions with initial data in $U_{\epsilon,\alpha}$, $0 < \alpha < \alpha'$, will always remain in a neighbourhood $U_{\epsilon,c'\sqrt{\epsilon}+\alpha}$, where the constants c', α' are independent of ϵ.

3. If $A > 0$ and $A_k \neq 0$, $k = 1, \ldots, n-2$; $n \geq 3$, then, as ϵ goes through zero, the system undergoes

a soft loss of metric stability: for small $\epsilon \geqslant 0$, there exist sets G_ϵ^s such that solutions with initial data in $G_\epsilon^s \cap U_{\epsilon,\alpha}$ will always remain in a neighbourhood $U_{\epsilon, c'\sqrt{\epsilon+\alpha}}$, where

$$\lim_{(\epsilon+\alpha)\to 0} \mu(G_\epsilon^s \cap U_{\epsilon,\alpha})/\mu(U_{\epsilon,\alpha}) = 1$$

and the constant c' is independent of ϵ.

Corollary. Under the assumptions of Theorem 1, if $n = 2$, $\epsilon = 0$ and $A > 0$, the position of equilibrium is Lyapunov stable if $n = 3$, $A > 0$ and $A_k \neq 0$ $(k = 1, \ldots, n-2)$, the position of equilibrium $x(0)$ is metrically stable.

Informally speaking, Theorem 1 states that in bifurcations of the type described above there are two possible destabilization scenarios. If $A < 0$ stability is lost all at once: when $\epsilon \geqslant 0$ solutions near the equilibrium position leave any neighbourhood of the order of unity. If $A > 0$ the instability develops gradually and is practically negligible at small $\epsilon > 0$.

Remarks. 1. The term "metric stability" was introduced by V. I. Arnol'd [2]. The earliest studies of the effects that we call "hard" and "soft" loss of stability in positions of equilibrium of systems of differential equations that depend on parameters are due to Poincaré and Andronov.

2. It has been proved [3] that a position of equilibrium with $n = 2$, $\epsilon = 0$, $A > 0$ is formally stable, and that the position $x(0)$ is unstable in the case $n = 2$, $A < 0$.

3. It has been shown [4, 5] that the position of equilibrium with $n = 2$, $\epsilon = 0$, $A > 0$ is Lyapunov stable; however, the proof in [4] involves an error—the integral defining the action variable is not evaluated correctly.

4. It is probably possible to put $c = 1$ in (1.4).

Let us consider the planar circular restricted three-body problem. In non-dimensional variables, the Hamiltonian depends on a single parameter—the mass of Jupiter $0 \leqslant \mu \leqslant \frac{1}{2}$. For every μ, the system has a position of equilibrium—a triangular point of libration. It is known that for $0 \leqslant \mu \leqslant \mu_1$, $\mu_1(1-\mu_1) = 1/27$, $\mu_1 = 0.03852\ldots$, the position of equilibrium is stable in the linear approximation. Moreover, for all but two values of μ in the interval $(0, \mu_1)$, the triangular points of libration are Lyapunov stable [6]. As the parameter μ passes through the value μ_1, the position of equilibrium bifurcates as described here [7]. Define $\epsilon = \mu - \mu_1$ and apply Theorem 1. In this case $n = 2$. Since the characteristic polynomial is $\lambda^4 + \lambda^2 + 27\mu(1-\mu)/4$, it follows that $d/d\epsilon|_{\epsilon=0} D_\epsilon = -27(1-2\mu_1) < 0$. It has been shown [7] that $A = 0.603\ldots > 0$. Thus, as μ goes through the critical value μ_1, the triangular point of libration undergoes a soft loss of stability.

2. PERIODIC SOLUTIONS

The stability of a periodic solution in the linear approximation is determined by the structure of the spectrum S^* of the monodromy matrix. In Hamiltonian systems the set $S^* \subset \mathbf{C}$ is symmetrical about the real axis and the unit circle; i.e. for any $\mu \in S^*$ the numbers $\bar{\mu}$, μ^{-1} and $\bar{\mu}^{-1}$ are also in S^*. The elements of S^* are the roots of the characteristic polynomial $f^*(\mu)$ and are known as multipliers. Unity is always not less than a double root of f^*.

Suppose we have a one-parameter family of autonomous Hamiltonian systems with n degrees of freedom phase space M and Hamiltonians H_ϵ^*; ϵ is a parameter taking values in a neighbourhood of zero. Let $\gamma(\epsilon)$ be a family of periodic solutions which is smooth with respect to ϵ.

Even in an individual autonomous Hamiltonian system, the periodic solutions form families; but since such families are generally parameterized by the energy, this case is easily reduced to the previous one (families $\gamma(\epsilon)$, H_ϵ^*).

Let the multipliers of a periodic solution $\gamma(0)$ have the form $\exp(\pm 2\pi i\omega_1), \ldots, \exp(\pm 2\pi i\omega_n)$, where $\omega_1, \ldots, \omega_n$ are real, $\omega_n = 1$. Let us assume that $\omega_1 = \omega_2 = \omega$ and for any $3 \leqslant l$, $m \leqslant n-1$

$$\omega \pm \omega_l \notin \mathbf{Z}, \quad \omega_m \pm \omega_l \notin \mathbf{Z}, \quad 2\omega \notin \mathbf{Z}, \quad 2\omega_l \notin \mathbf{Z}$$

Suppose that as ϵ goes through zero two pairs of multipliers $\gamma(\epsilon)$, lying on the unit circle in the plane $\mathbf{C}$, collide at points $\exp(\pm 2\pi i\omega)$ and leave the circle, becoming a quadruple of the form $\exp(\pm\alpha \pm i\beta)$, $\alpha \in \mathbf{R}\backslash\{0\}$, $\beta/\pi \in \mathbf{R}\backslash\mathbf{Z}$. Thus, for negative ϵ of small absolute value the periodic solutions $\gamma(\epsilon)$ are orbitally stable in the linear approximation; when the parameter goes through zero they lose stability.

Any relation of the type

$$k\omega + k_3\omega_3 + \ldots + k_n\omega_n = 0 \tag{2.1}$$

where $k, k_3, \ldots, k_n$ are integers, will be called a resonance of order $|k|+|k_3|+\ldots+|k_{n-1}|$.

The characteristic polynomials $f_\epsilon^*(\mu)$ of periodic solutions $\gamma(\epsilon)$ are reciprocal, i.e. $f_\epsilon^*(\mu) = \mu^{2n} f_\epsilon^*(\mu^{-1})$. In addition, $f_\epsilon^*(1) = 0$. Consequently, the equation $f_\epsilon^*(\mu) = 0$ is equivalent to the equation $(\mu-1)^2\varphi_\epsilon(\mu+\mu^{-1}) = 0$, where φ_ϵ is a polynomial of degree $n-1$. Let D_ϵ^* denote the discriminant of φ_ϵ.

Let $U_{\epsilon,\alpha}$, $\alpha>0$, be a family of neighbourhoods of the form

$$U_{\epsilon,\alpha} = \{ x \in M:\ \mathrm{dist}(x, \gamma(\epsilon)) < \alpha \}$$

where the distance from a point x to the curve $\gamma(\epsilon)$ is understood in some Riemannian metric on M.

Lemma 2. Assume that the frequencies $\omega, \omega_3, \ldots, \omega_n$ do not satisfy resonances of orders less than or equal to four and $d/d\epsilon|_{\epsilon=0} D_\epsilon^* \neq 0$. Then, for each ϵ in some neighbourhood of zero, one can introduce local canonical coordinates p, q, y, x, s, ψ; $p = (p_1, p_2)$, $q = (q_1, q_2)$, $y = (y_1, \ldots, y_{n-3})$, $x = (x_1, \ldots, x_{n-3})$, $\psi = \psi \bmod 2\pi$, such that

(a) the pairs of variables p and q, y and x, s and ψ are canonically conjugate;

(b) the curve $\gamma(\epsilon)$ is defined by the equations $p = q = 0$, $y = x = 0$, $s = 0$;

(c) the following relationships hold:

$$H_\epsilon^* = \frac{2\pi}{T(\epsilon)}\left(-s + G(p, q, \epsilon) + \sum_{k=1}^{n-3} G_k(y_k, x_k, \epsilon)\right) + G^*$$

$$G^* = \epsilon O_4 + O_5 + O(s^2) + sO_2, \quad O_l = O_l(p, q, y, x)$$

where the functions G and G_k are given by (1.2), (1.3) and $T(\epsilon)$ is the period of $\gamma(\epsilon)$.

Remarks. 1. Physical considerations dictate that $n \geqslant 3$. If $n = 3$ the functions G_k and variables y, x do not appear in the Hamiltonian H_ϵ^*.

2. If $A \neq 0$ and $A_k \neq 0$, $1 \leqslant k \leqslant n-2$, then for small negative ϵ of small absolute value the trajectories $\gamma(\epsilon)$ are orbitally metrically stable in the sense of Remark 6 to Lemma 1.

Theorem 2. Under the assumptions of Lemma 2, suppose that as the parameter ϵ is varied a periodic trajectory γ of an analytic Hamiltonian system undergoes a bifurcation of the type described at the beginning of Sec. 2. Then the following statements hold.

1. If the constant A in the Hamiltonian G is negative then, as the parameter ϵ goes through zero, the system undergoes a hard loss of metric orbital stability: for any small $\epsilon \geqslant 0$, there exists a set $G_\epsilon^u \subset M$ such that, for any $\alpha>0$, solutions with initial data in $G_\epsilon^u \cap U_{\epsilon,\alpha}$ will leave a neighbourhood U_{ϵ,α_0}, where $\alpha_0>0$ is independent of ϵ, α, and for $\alpha<c^*$ we have $\mu(G_\epsilon^u \cap U_{\epsilon,\alpha})/\mu(U_{\epsilon,\alpha}) \geqslant c > 0$.

2. If $A>0$ and $A_k \neq 0$ $(k = 1, \ldots, n-2)$ then, as ϵ goes through zero, the system undergoes a soft loss of stability; for small $\epsilon \geqslant 0$, there exist sets G_ϵ^s such that solutions with initial data in $G_\epsilon^s \cap U_{\epsilon,\alpha}$ will always remain in a neighbourhood $U_{\epsilon,c'\sqrt{\epsilon+\alpha}}$, where

$$\lim_{(\epsilon+\alpha)\to 0} \mu(G_\epsilon^s \cap U_{\epsilon,\alpha})/\mu(U_{\epsilon,\alpha}) = 1$$

and the constant c' is independent of ϵ.

Let us consider a triangular solution of the planar elliptic restricted three-body problem. The problem has two parameters: the mass of Jupiter μ and the eccentricity e of elliptic motion in the Sun–Jupiter system; the Hamiltonian H is a periodic function of time t. The problem may be reduced to an autonomous problem with three degrees of freedom if we consider t as a phase variable, introduce momentum s as a variable canonically conjugate to t and a new time variable τ, and consider the Hamiltonian H–s.

Numerical techniques have been used [8] to construct the domain of orbital stability of the triangular solution in the linear approximation in the rectangle $\{(\mu, e)\colon 0 \leqslant \mu \leqslant 1/2, 0 \leqslant e \leqslant 1\}$. One of the components Γ of the boundary of this domain begins at the point $(\mu, e) = (\mu_1, 0)$ (see Sec. 1) and enters the domain of positive e values. As the parameters vary along the straight line $e = 0$ the system leaves the stability domain—this was discussed in Sec. 1. At $e = 0$ the triangular solution becomes a point of libration. If one nevertheless considers points of libration as 2π-periodic solutions, then as the parameter μ goes through μ_1 the system undergoes a bifurcation as described in Sec. 2.

Let us verify that the assumptions of Theorem 2 are satisfied. Since the frequency ω is found from the equation $\omega^4 - \omega^2 + 1/4 = 0$ (see Sec. 1) and is equal to $\pm 1/\sqrt{2}$, there are no resonances (2.1). That $dD^*_\epsilon/d\epsilon(0) \neq 0$ follows from the fact that $dD_\epsilon/d\epsilon(0) \neq 0$. The inequality $A > 0$ was also discussed in Sec. 1. Consequently, the assumptions of Theorem 2 are satisfied if $e = 0$.

Let us assume now that the system leaves the stability domain not along the straight line $e = 0$, but along some curve $\sigma = \{\mu(\epsilon), e(\epsilon)\}$ which cuts Γ transversally at $\epsilon = 0$, in a point close to $(\mu_1, 0)$. By continuity, the assumptions of Theorem 2 will continue to hold. Thus, as the parameters μ, e vary along σ, the triangular solution of the restricted planar elliptic three-body problem undergoes a soft loss of metric orbital stability at $\epsilon = 0$.

3. NORMAL FORM OF FAMILY H_ϵ

We shall now prove Lemma 1 and the statements made in Remarks 4–6 thereafter. The proof of Lemma 2 is analogous, using normal forms of the Hamiltonians in the neighbourhood of a periodic solution [9].

We begin the proof of Lemma 1 with the case $\epsilon = 0$. We may assume that the canonical local coordinates on M have already been introduced, in such a way that the equilibrium positions $x(\epsilon)$ are situated at zero. Then the Maclaurin expansion of H_ϵ begins with quadratic terms (it may be assumed that $H_\epsilon(0) = 0$). The quadratic part of H_ϵ at $\epsilon = 0$ may be written as follows [10]:

$$H_0^{(2)} = \omega(p_2 q_1 - p_1 q_2) + 2p^2 + \sum_{k=1}^{n-2} \omega_{k+2} \frac{y_k^2 + x_k^2}{2}$$

Using versal deformations $H_0^{(2)}$ in the class of quadratic Hamiltonians [1, 11], we obtain the canonical form of the quadratic part of H_ϵ:

$$H_\epsilon^{(2)} = H_0^{(2)} + \epsilon\omega'(\epsilon)(p_2 q_1 - p_1 q_2) - \epsilon g(\epsilon) q^2 + \sum_{k=1}^{n-2} \epsilon\omega'_{k+2}(\epsilon) \frac{y_k^2 + x_k^2}{2}$$

The condition $g(0) > 0$ follows from the stability of the equilibrium at $\epsilon < 0$ and the inequality $dD_\epsilon/d\epsilon(0) \neq 0$.

Using the fact that there are no low-order resonances among the frequencies $\omega, \omega_2, \ldots, \omega_n$ and the Normal Form Theorem for Hamiltonians depending on a parameter [12], we reduce H_ϵ to the form

$$H_\epsilon = H_\epsilon^{(2)} + H^{(4)}(p, q) + \sum_{k=1}^{n-2} A_k (y_k^2 + x_k^2)^2 + \epsilon O_4 + O_5$$

where $H^{(4)}$ is a homogeneous form of degree 4.

Finally, a canonical change of variables, leaving y, x unchanged and affecting only p and q, reduces the Hamiltonians to the form of (1.1) (see [3]). This completes the proof of the lemma.

The assertion of Remark 4 is proved as follows. The change of variables

$$p' = \alpha p, \quad q' = \alpha q, \quad y' = \alpha y, \quad x' = \alpha x, \quad \alpha = \begin{cases} 1, & \text{if } A = 0 \\ |8A|^{-1/2}, & \text{if } A \neq 0 \end{cases}$$

leaves the system Hamiltonian, and in fact the new Hamiltonian

$$H'_\epsilon(p', q', y', x', \epsilon) = \alpha^2 H_\epsilon(p, q, y, x, \epsilon)$$

will have the same form as H_ϵ. However, the coefficients A_k, B, C in the functions G_k, G are multiplied by a positive constant and A is replaced by $\operatorname{sgn} A/8$.

The assertion of Remark 5 follows from a theorem on the stability of positions of equilibrium of Hamiltonian systems with two degrees of freedom [2, 13].

The metric stability in Remark 6 follows from the fact that, in a small neighbourhood of zero, for small $\epsilon<0$, the system with Hamiltonian H_ϵ has a large number of invariant tori [2].

4. BIFURCATION OF EQUILIBRIUM POSITIONS

We will now outline the main steps in the proof of Theorem 1. We will begin with part 1.

Consider the Lyapunov function

$$V = p_1q_1 + p_2q_2 - \sum_{k=1}^{n-2}(y_k^2 + x_k^2)$$

Since

$$dV/dt = 4p^2 + 2q^2[-2Aq^2 + \epsilon g(\epsilon) + B(p_2q_1 - p_1q_2) + 2Cp^2] + \epsilon O_4 + O_5$$

it follows that $dV/dt>0$ for $A<0$ and small $\epsilon\geqslant 0$ in $U^+_{\epsilon,\alpha} = U_{\epsilon,\alpha}\cap\{V>0\}$. Thus V will increase along a trajectory with initial data in $U^+_{\epsilon,\alpha}$, as long as the trajectory remains in $U_{\epsilon,\alpha}$. In addition, $\mu(U^+_{\epsilon,\alpha})/\mu(U_{\epsilon,\alpha})\geqslant c>0$, since the sets $\{V>0\}$ and $U_{\epsilon,\alpha}$ are the interior of a cone and a sphere, respectively, with identical centres of symmetry.

To prove parts 2 and 3 of Theorem 1, we first transform H_ϵ, by making the following change of variables one after the other:

$$\begin{aligned}
&q = \delta^{1/2}q',\ \ p = \delta p',\ \ x = \delta^{3/4}\sqrt{2\eta}\cos\xi,\ \ y = \delta^{3/4}\sqrt{2\eta}\sin\xi\\
&q_1' = p_1'' - q_2'',\ \ q_2' = p_2'' - q_1'',\ \ p_1' = (p_2'' + q_1'')/2,\ \ p_2' = (p_1'' + q_2'')/2\\
&p_k'' = \sqrt{2r_k}\sin\varphi_k,\ \ q_k'' = \sqrt{2r_k}\cos\varphi_k,\ \ k = 1, 2\\
&\rho_1 = r_1,\ \ \rho_2 = -r_1 + r_2,\ \ \psi_1 = \varphi_1 + \varphi_2,\ \ \psi_2 = \varphi_2
\end{aligned} \tag{4.1}$$

(the last three changes leave η, ξ invariant).

As a result we obtain a Hamiltonian system with Hamiltonian $H_{\epsilon,\delta}(\rho, \psi, \eta, \xi) = \delta^{-3/2}H_\epsilon(p, q, y, x)$.

Before explicitly writing out $H_{\epsilon,\delta}$, we will introduce some assumptions.

1. In what follows we will assume that the parameter δ is confined to the interval

$$0\leqslant 2\epsilon\leqslant\delta\leqslant\delta_0 \tag{4.2}$$

where $\delta_0>0$ is sufficiently small.

2. We assume throughout that $g(\epsilon)\equiv 1$; otherwise, we need only substitute $\epsilon\to\epsilon g(\epsilon)$.

3. In view of the inequality $A>0$ and Remark 4 to Lemma 1, we may assume that $A = 1/8$.

Thus, we have

$$H_{\epsilon,\delta} = F_0(\rho_2, \eta, \epsilon) + \delta^{1/2}F_1(\rho, \psi_1, \frac{\epsilon}{\delta}) + \delta F_2(\rho, \psi, \frac{\epsilon}{\delta}, \delta) + \delta^{3/2}F_3(\eta) + \\ + \delta^2F_4(\rho, \psi, \eta, \xi, \frac{\epsilon}{\delta}, \delta)$$

$$F_0 = -\omega\rho_2 + \sum_{k=1}^{n-2}(\omega_{k+2} + \epsilon\omega'_{k+2}(0))\eta_k$$

$$F_1 = (1 - \frac{\epsilon}{\delta})(2\rho_1 + \rho_2) + 2(1 + \frac{\epsilon}{\delta})\sqrt{\rho_1(\rho_1 + \rho_2)}\sin\psi_1 + \\ \frac{1}{2}(2\rho_1 + \rho_2 - 2\sqrt{\rho_1(\rho_1 + \rho_2)}\sin\psi_1)^2$$

$$F_3 = \sum_{k=1}^{n-2}4A_k\eta_k^2$$

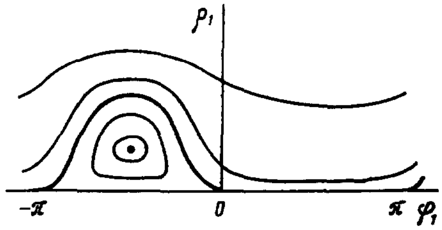

FIG. 1.

The Hamiltonian $H_{\epsilon,\delta}$ is an analytic function of the variables

$$\sqrt{\rho_1},\ \sqrt{\rho_1+\rho_2},\ \sqrt{\eta_1},\ldots,\sqrt{\eta_{n-2}},\ \psi,\ \xi.$$

Up to terms of order δ, the system is a composition of a fast (unperturbed) system and a slow one (with Hamiltonian F_1). The analysis of F_1 requires cumbersome arguments.

The main results are as follows. The Hamiltonian F_1 depends on ρ_2; we may assume that $-1\leqslant\rho_2\leqslant 1$. In addition, since

$$F_1(\rho_1,\rho_2,\psi_1,\epsilon/\delta)=F_1(\rho_1+\rho_2,-\rho_2,\psi_1,\epsilon/\delta)$$

we may confine our attention to values of ρ_2 in $[0, 1]$. The level curves of F_1 for fixed $\rho_2\in[0,1]$, $\epsilon/\delta\in[0,1/2]$ (the phase portrait of the slow system) are shown in Fig. 1.

The phase space $\{(\psi_1,\rho_1):\rho_1\geqslant 0,\ \psi_1 \bmod 2\pi\}$ is a closed half-cylinder. All non-empty non-critical levels $F_1=\text{const}$ are connected and diffeomorphic to circles. For each assignment of parameter values, F_1 has a minimum on the axis $\psi_1=-\pi/2$. Moreover, there is yet another critical level $F_1=F_*\equiv(1-\epsilon/\delta)\rho_2+\rho_2^2/2$, which is the union of the circle $\rho_1=0$ and a singular curve that touches the circle at the points $(-\pi,0)$ and $(0,0)$ (see Fig. 1). Since ψ_1, ρ_1 are in fact polar coordinates, the circle $\{\rho_1=0,\ \psi_1 \bmod 2\pi\}$ may be considered as a single point, so that the level $F_1=F_*$ is also an invariant circle of the slow system. We now define the action variable as

$$I_1(f,\rho_2,\frac{\epsilon}{\delta})=\frac{1}{2\pi}\int_{\{F_1=f\}}\rho_1 d\psi_1\geqslant 0$$

Let $\Phi(\cdot,\rho_2,\epsilon/\delta)$ be the function inverse to $I_1(\cdot,\rho_2,\epsilon/\delta)$ for fixed values of ρ_2, ϵ/δ. Essentially, Φ is just the slow Hamiltonian expressed in terms of I_1, ρ_2, ϵ/δ.

Let Z_α, $\alpha>0$, be a set of the form $\{(\psi_1,\rho_1): 0\leqslant\rho_1\leqslant\alpha\}$.

Lemma 3. Positive constants c_*, c_2, c_3, c_3', c_4 exist such that for any $-1\leqslant\rho_2\leqslant 1$, $0\leqslant\epsilon/\delta\leqslant 1/2$:

1. The subset $X=\{0\leqslant I_1\leqslant c_2\}$ of the phase half-cylinder satisfies the condition $Z_1\subset X\subset Z_{c*}$.
2. For any $I_1\in[0,c_2]$,

$$c_3\leqslant\partial\Phi/\partial I_1\leqslant c_3',\quad 0\not\equiv|\partial^2\Phi/\partial I_1^2|\leqslant c_4$$

Let I_1, ϑ_1 be action-angle variables in the system with Hamiltonian F_1. The transformation ρ_1, $\psi_1\to I_1$, ϑ_1 is given by a generating function $S(\rho_1,\vartheta_1,\rho_2,\epsilon/\delta)$. The change of variables ρ_1, ρ_2, ψ_1, $\psi_2\to I_1$, I_2, ϑ_1, ϑ_2, by means of the generating function $\rho_2\vartheta_2+S$, defines action-angle variables in the system with Hamiltonian $-\omega\rho_2+\sqrt{\delta}F_1$. Under these conditions $I_2=\rho_2$. The Hamiltonian $H_{\epsilon,\delta}$ in variables I, ϑ, η, ξ becomes

$$H'_{\epsilon,\delta}=F_0(I_2,\eta,\epsilon)+\delta^{1/2}\Phi(I,\frac{\epsilon}{\delta})+\delta F_2'(I,\vartheta,\frac{\epsilon}{\delta},\delta)+\delta^{3/2}F_3(\eta)+\delta^2F_4'(I,\vartheta,\eta,\xi,\frac{\epsilon}{\delta},\delta)$$

A canonical change of variables leaving η, ξ unchanged will confine the dependence on the variable ϑ_2 to terms of order δ^2. The Hamiltonian $H'_{\epsilon,\delta}$ then becomes

$$H''_{\epsilon,\delta}=F_0(I_2',\eta,\epsilon)+\delta^{1/2}\Phi(I',\frac{\epsilon}{\delta})+\delta F_2''(I',\vartheta_1',\frac{\epsilon}{\delta},\delta)+\delta^{3/2}F_3(\eta)+\delta^2F_4''(I',\vartheta',\eta,\xi,\frac{\epsilon}{\delta},\delta)$$

We now adjust the variables I', ν' so that they become action-angle variables in the system with

Hamiltonian $-\omega I_2' + \sqrt{\delta}\Phi + \delta F''$. The original Hamiltonian written in terms of the new variables, will become

$$H'''_{\epsilon,\delta} = F_0(I_2'', \eta, \epsilon) + \delta^{1/2}\Phi'(I_2'', \frac{\epsilon}{\delta}, \delta) + \delta^{3/2} F_3(\eta) + \delta^2 F_4'''(I'', \vartheta'', \eta, \xi, \frac{\epsilon}{\delta}, \delta) \tag{4.3}$$

$$\Phi'(I, \epsilon/\delta, \delta) = \Phi(I, \epsilon/\delta) + O(\delta^{1/2}) \tag{4.4}$$

The system with Hamiltonian (4.3) is integrable to within $O(\delta^2)$. The unperturbed system (i.e. the system with Hamiltonian F_0) is degenerate and iso-energetically degenerate, but the iso-energetic degeneracy may be eliminated by terms $\sqrt{\delta}\Phi'$ and $\delta^{3/2}F_3$ for I'' values such that $\partial^2\Phi'/\partial I_1^2 \neq 0$. Since Φ' is analytic, almost all I'' satisfy this condition.

Let $V^{\beta}_{\epsilon,\delta}$ be a neighbourhood of the equilibrium position $x(\epsilon)$, of the following form:

$$V^{\beta}_{\epsilon,\delta} = \{(\rho, \psi, \eta, \xi): \ |\rho_l| < \beta^2/2, \ |\eta_k| < \beta^2/2, \ l = 1, 2; \ k = 1, \ldots, n-2\}$$

and let $W^{\beta}_{\epsilon,\delta}$ be the closure of the set of points in the phase space that lie on invariant roti $\mathbf{T}_\nu$ of the system with Hamiltonian $F_0 + \delta^{1/2}\Phi' + \delta^{3/2}F_3$ such that $\mathbf{T}_\nu \cap V^{\beta}_{\epsilon,\delta} \neq \phi$. Obviously, there is a constant $c_s > 0$ for which

$$V^1_{\epsilon,\delta} \subset W^1_{\epsilon,\delta} \subset V^{c_s}_{\epsilon,\delta} \tag{4.5}$$

In addition, analysis of the change of variables (4.1) $p, q, y, x \to \rho, \psi, \eta, \xi$ gives

$$V^{c_6\alpha/\sqrt{\delta}}_{\epsilon,\delta} \subset U_{\epsilon,\alpha} \subset V^{c_7\alpha/\delta}_{\epsilon,\delta}; \quad c_6, c_7 > 0, \quad 0 < \alpha < \text{const}. \tag{4.6}$$

Let L_h be the energy level $\{H'''_{\epsilon,\delta} = h\}$ and μ_h the measure induced on L_h by some metric of the phase space, such as

$$\sum_{k=1}^{2} (dI_k^2 + d\vartheta_k^2) + \sum_{s=1}^{n-2} (d\eta_s^2 + d\xi_s^2)$$

Lemma 4. Functions $d_k(\delta) \geq 0$, $\lim d_k(\delta) = 0$ exist as $\delta \to 0$, $k = 1, 2$, such that for any ϵ, δ satisfying conditions (4.2) the majority of tori $\mathbf{T}_\nu \subset W^1_{\epsilon,\delta}$ do not disintegrate, but are slightly deformed into invariant tori $\mathbf{T}_\nu(\epsilon, \delta)$ of the perturbed system [i.e. the system with Hamiltonian (4.3)], by a deformation of at most $d_1(\delta)$; that is to say, $\mathbf{T}_\nu(\epsilon, \delta)$ can be derived from $\mathbf{T}_\nu$ by subjecting all its points to a translation by at most $d_1(\delta)$. Moreover, for $h \in [-\omega/4, \omega/4]$, the measure of the set $X_{\epsilon,\delta} \subset V_{\epsilon,\delta} \cap L_h$ of all points not situated on invariant tori of the unperturbed system is at most $d_2(\delta)\mu(V_{\epsilon,\delta} \cap L_h)$.

The proof of Lemma 4 uses standard techniques of KAM-theory. Similar results have been proved before (see, e.g. [14]). The proof relies on uniform estimates for the derivatives of Φ', similar to those obtained in Lemma 3; these estimates follow easily from (4.4). One also uses the analyticity of the Hamiltonian $H'''_{\epsilon,\delta}$ and the estimate $|F_4''| < c_5$ in $V^{c_5}_{\epsilon,\delta}$. One obstacle in the proof is the fact that the derivative $\partial^2\Phi'/\partial I_1^2$ may vanish at certain points. One must therefore first prove, fixing $\gamma > 0$, that tori $\mathbf{T}_\nu$ for which $|\partial^2\Phi'/\partial I_1^2| \geq \gamma$ are preserved, and then let γ go to zero. Iso-energetic reduction effects the passage to the energy level L_h. The details of this proof will be omitted.

Let $0 < \epsilon < \delta_0/2$ and $0 < \alpha < (\delta_0 - 2\epsilon)/c_9$, where c_9 is a positive constant, $c_0 \geq \max\{c_7, 2\}$. Set $\delta = c_9(\alpha + \epsilon)$. Then, obviously ϵ, δ will satisfy conditions (4.2). By (4.6),

$$U_{\epsilon,\alpha} \subset V^{c_7\alpha/\delta}_{\epsilon,\delta} \subset V^{c_7/c_9}_{\epsilon,\delta} \subset V^1_{\epsilon,\delta}$$

Moreover, if c_9 is sufficiently large, then $|H'''_{\epsilon,\delta}| < \omega/4$ in the neighbourhood $C^{c_7/c_9}_{\epsilon,\delta}$ and we can apply Lemma 4.

Part 2 of Theorem 1 may now be proved as follows. Take δ_0 so small that $d_k(\delta) < d_k(\delta_0) < 1/2$, $k = 1, 2$. Then the surfaces L_h that intersect the neighbourhood $V^{c_7/c_9}_{\epsilon,\delta}$ will contain "many" invariant tori $\mathbf{T}_\nu(\epsilon, \delta)$. Since the number n of degrees of freedom is 2, the tori $\mathbf{T}_\nu(\epsilon, \delta)$ divide the energy levels L_h. If c_9 is large enough, any point of $\mathbf{V}^{c_7/c_9}_{\epsilon,\delta}$ will lie inside one of the tori $\mathbf{T}_\nu(\epsilon, \delta)$. By Lemma 4 and the inclusion relation (4.5), $\mathbf{T}_\nu(\epsilon, \delta) \subset V^{c_5+1/2}_{\epsilon,\delta}$. Consequently, solutions with initial data in $U_{\epsilon,\alpha}$ will not leave the neighbourhood

Lemma 4 and the inclusion relation (4.5), $\mathbf{T}_\nu(\epsilon, \delta) \subset V_{\epsilon,\delta}^{c_5+1/2}$. Consequently, solutions with initial data in $U_{\epsilon,\alpha}$ will not leave the neighbourhood

$$V_{\epsilon,\delta}^{c_5+1/2} \subset U_{\epsilon,(c_5+1/2)\sqrt{\delta}/c_6} = U_{\epsilon, c\sqrt{\alpha+\epsilon}}, \quad c = (c_5 + 1/2)\sqrt{c_9}/c_6$$

as required.

The proof of part 3 of Theorem 1 is similar. We consider an arbitrary small $\gamma > 0$, and take δ_0 so that $d_k(\delta) < d_k(\delta_0) < \gamma$, $k = 1, 2$. Then the neighbourhood $U_{\epsilon,\alpha}$ will contain a set of measure greater than $(1-\gamma)\mu(U_{\epsilon,\alpha})$, each of whose points lies on an invariant torus $\mathbf{T}_\nu(\epsilon,\delta) \subset U_{\epsilon,c\sqrt{\alpha+\epsilon}}$.

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